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Excited Binomial States and Excited Negative Binomial States of the Radiation Field and Some of Their Statistical Properties

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We introduce excited binomial states and excited negative binomial states of the radiation field by repeated application of the photon creation operator on binomial states and negative binomial states. They reduce to Fock states and excited coherent states in certain limits and can be viewed as intermediate states between Fock states and coherent states. We find that both the excited binomial states and excited negative binomial states can be exactly normalized in terms of hypergeometric functions. Base on this interesting characteristic, some of the statistical properties are discussed.

1. INTRODUCTION

The interesting nonclassical states which are engendered by excitations on particular quantum states have been examined. These states were introduced by Agarwal and Tara as the excited coherent states (ECS) [1]. The ECS exhibit remarkable nonclassical properties such as sub-Poisonnian photon statistics and squeezing in one of the quadratures of the radiation field, etc. Several other excited quantum states, even and odd ECS [2], excited squeezed states [3–6], and excited thermal states [7, 8] have been discussed in the literature. It was shown that some of the excited quantum states can be prepared in the processes of the field–atom interaction in a cavity [1] or via conditional measurement on a beam splitter [9].

In this work, we introduce the excited binomial states (EBS) and excited negative binomial states (ENBS) of the radiation field by repeated application

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of the photon creation operator on binomial states (BS) [10–15] and negative binomial states (NBS) [16–22]. We examine the mathematical and physical properties of such states. They interpolate between Fock states and coherent states in the sense that they reduce to Fock states and coherent states in different limits. An interesting property is found that both the EBS and ENBS are exactly normalized in terms of hypergeometric functions [23]. These are demonstrated in Section 2. In Section 3, we study the sub-Poissonian statistics and squeezing effects of these states. The conclusion is given in Section 4.

2. EBS AND ENBS

We introduce the EBS $|k, \eta, M\rangle$ and ENBS $|k, \eta, M\rangle^-$ defined by

$$|k, \eta, M\rangle = \mathcal{N}(k, \eta, M)a^{\dagger k}|\eta, M\rangle \tag{1}$$

$$|k, \eta, M\rangle^- = \mathcal{N}^-(k, \eta, M)a^{\dagger k}|\eta, M\rangle^- \tag{2}$$

where

$$|\eta, M\rangle = \sum_{n=0}^{M} C_n(\eta, M)|n\rangle = \sum_{n=0}^{M} \binom{M}{n}^{1/2} \eta^n(1-\eta^2)^{(M-n)/2}|n\rangle \tag{3}$$

$$|\eta, M\rangle^- = \sum_{n=0}^{\infty} C_n^-(\eta, M)|n\rangle = \sum_{n=0}^{\infty} \binom{M+n-1}{n}^{1/2} \eta^n(1-\eta^2)^{M/2}|n\rangle \tag{4}$$

are the BS and NBS, respectively. Here k, n, and M are integers, η is a real number, $|n\rangle$ is a Fock state, $a^\dagger$ and its conjugate a are boson creation and annihilation operators, and $\mathcal{N}(k, \eta, M)$ and $\mathcal{N}^-(k, \eta, M)$ are normalization constants of the EBS and ENBS, respectively.

In two limits, $\eta \to 1$ and $\eta \to 0$, the BS reduce to the Fock state $|M\rangle = |1, M\rangle$ and vacuum state $|0\rangle = |0, M\rangle$, and the EBS to Fock states $|M + k\rangle$ and $|k\rangle$, respectively. In the different limit of $M \to \infty$, $\eta \to 0$ with $\eta^2 M = \alpha^2$ fixed (α is real), the BS reduce to coherent states $|\alpha\rangle$ and the EBS to the ECS. In the limit of $\eta \to 0$, the NBS reduce to the vacuum state and ENBS to the Fock state $|k\rangle$. When $M \to \infty$, $\eta \to 0$ with $\eta^2 M = \alpha^2$ fixed, the NBS reduce to coherent states and the ENBS to the ECS. Thus, both the EBS and ENBS can be reduced to the Fock states and ECS. Note that the ECS can be reduced to coherent states; the two excited states can be recognized as intermediate states between Fock states and coherent states.

In determining the normalization constant of the EBS, we calculate the expectation value of operator $a^k a^{\dagger k}$ on the BS,

$$\begin{aligned} B(k, \eta, M) &= \langle \eta, M|a^k a^{\dagger k}|\eta, M\rangle \\ &= M!\,(1-\eta^2)^M \sum_{n=0}^{M} \frac{(n+k)!}{n!n!(M-n)!}\left(\frac{\eta^2}{1-\eta^2}\right)^n \end{aligned}$$

$$= M!\,\eta^{2M}\sum_{n=0}^{M}\frac{(M+k-n)!}{n!(M-n)!(M-n)!}\left(\frac{1-\eta^2}{\eta^2}\right)^n$$

$$= \eta^{2M}[(M+k)!/M!]\ {}_2F_1(-M, -M; -M-k; (\eta^2-1)/\eta^2) \quad (5)$$

where ${}_2F_1\,(\alpha, \beta; \gamma; x)$ is the hypergeometric function [23]. The normalization constant is $\mathcal{N}(k, \eta, M) = 1/\sqrt{B(k, \eta, M)}$.

Using the same procedure as above, we give the expectation value of operator $a^k a^{\dagger k}$ on NBS

$$B^-(k, \eta, M) = {}^-\langle \eta, M | a^k a^{\dagger k} | \eta, M\rangle^-$$

$$= \frac{(1-\eta^2)^M}{(M-1)!}\sum_{n=0}^{\infty}\frac{(M+n-1)!(n+k)!}{n!n!}\,\eta^{2n} \quad (6)$$

It is an infinite sum. The normalization constant of the ENBS is $\mathcal{N}^-(k, \eta, M) = 1/\sqrt{B^-(k, \eta, M)}$.

Now we give a different form of $B^-(k, \eta, M)$ as a finite sum. The normal ordering of the operator $a^k a^{\dagger k}$ is

$$a^k a^{\dagger k} = \sum_{l=0}^{k}\frac{k!k!a^{\dagger(k-l)}a^{k-l}}{l!(k-l)!(k-l)!} \quad (7)$$

It is easy to evaluate from Eq. (4) that

$$a^k|\eta, M\rangle^- = \left(\frac{\eta}{\sqrt{1-\eta^2}}\right)^k\sqrt{\frac{(M+k-1)!}{(M-1)!}}\,|\eta, M+k\rangle^- \quad (8)$$

$${}^-\langle \eta, M|a^{\dagger k}a^k|\eta, M\rangle^- = \left(\frac{\eta^2}{1-\eta^2}\right)^k\frac{(M+k-1)!}{(M-1)!} \quad (9)$$

From Eqs. (7) and (9), we have

$$B^-(k, \eta, M) = \sum_{l=0}^{k}\frac{k!k!(M+k-l-1)!}{l!(k-l)!(k-l)!(M-1)!}\left(\frac{\eta^2}{1-\eta^2}\right)^{k-l} \quad (10)$$

The above equation can be written in terms of hypergeometric functions as

$$B^-(k, \eta, M) = \left(\frac{\eta^2}{1-\eta^2}\right)^k\frac{(M+k-1)!}{(M-1)!} \times {}_2F_1\left(-k,-k; -M-k+1; \frac{\eta^2-1}{\eta^2}\right) \quad (11)$$

It is interesting that the normalization constants of both the EBS and

ENBS can be expressed in terms of hypergeometric functions. The expressions are useful in the following investigations of nonclasscal properties of the two excited states.

3. NONCLASSICAL PROPERTIES

We expand the EBS and ENBS in terms of Fock states as

$$|k,\eta,M\rangle = \sum_{n=k}^{M+k} D_n(k,\eta,M)|n\rangle = \mathcal{N}(k,\eta,M)\sum_{n=k}^{M+k} C_{n-k}(\eta,M)\sqrt{\frac{n!}{(n-k)!}}\,|n\rangle \tag{12}$$

$$|k,\eta,M\rangle^{-} = \sum_{n=k}^{\infty} D_n^{-}(k,\eta,M)|n\rangle = \mathcal{N}^{-}(k,\eta,M)\sum_{n=k}^{\infty} C_{n-k}^{-}(\eta,M)\sqrt{\frac{n!}{(n-k)!}}\,|n\rangle \tag{13}$$

It can be seen that the Fock states $|0\rangle, |1\rangle, \ldots, |k-1\rangle$ are removed from the BS and NBS.

3.1. Photon Statistics

The mean photon number of the EBS is given by

$$\langle k,\eta,M|a^\dagger a|k,\eta,M\rangle = \langle k,\eta,M|aa^\dagger|k,\eta,M\rangle - 1 = \frac{B(k+1,\eta,M)}{B(k,\eta,M)} - 1 \tag{14}$$

The mean value of the operator $(a^\dagger a)^2$ can be obtained by expressing it in the anti-normally ordered form $(a^\dagger a)^2 = a^2a^{\dagger 2} - 3aa^\dagger + 1$ as

$$\langle k,\eta,M|(a^\dagger a)^2|k,\eta,M\rangle = \frac{B(k+2,\eta,M)}{B(k,\eta,M)} - 3\,\frac{B(k+1,\eta,M)}{B(k,\eta,M)} + 1 \tag{15}$$

Mandel's Q parameter defined by

$$Q = \frac{\langle (a^\dagger a)^2\rangle - \langle a^\dagger a\rangle^2}{\langle a^\dagger a\rangle} - 1 \tag{16}$$

measures the deviation from the Poisson distribution (the coherent states, $Q = 0$). If $Q < 0$ (> 0), the field is called sub (super)-Poissonian, respectively.

From Eqs. (14)–(16), we can easily derive Mandel Q parameter

$$Q = [B(k+2, \eta, M)B(k, \eta, M) - B(k+1, \eta, M)B(k, \eta, M) - B(k+1, \eta, M)^2]/[B(k+1, \eta, M)B(k, \eta, M) - B(k, \eta, M)^2] - 1 \tag{17}$$

Let $B \to B^-$ in Eqs. (14), (15), and (17); we can obtain the expectation values of $a^\dagger a$ and $(a^\dagger a)^2$ on the ENBS and the corresponding Q parameter.

In Fig. 3 we plot the Q parameter of the EBS as a function of η for different values of k. For $k = 0$, the EBS become the BS and $Q = -\eta^2$ [13]. The Q parameters of the EBS $|k, 0, M\rangle$ and $|k, 1, M\rangle$ are equal to -1 for $k \neq 0$. This is easily understood, for the states $|k, \eta, M\rangle$ reduce to Fock states $|k\rangle$ and $|k + M|$ in the limits $\eta \to 0$ and $\eta \to 1$ and the Q parameter of Fock states $|n\rangle$ ($n \neq 0$) are equal to -1. We see that the field in the EBS exhibits a significant amount of sub-Poissonian statistics. The Q parameters of the ENBS are displayed in Fig. 2 as a function of η for different values of k. For $k = 0$, the ENBS becomes the NBS. The Q parameter of the NBS is equal to $\eta^2/(1 - \eta^2)$ [20] and the NBS always show super-Poissonian statistics except for $\eta = 0$. For $k \neq 0$ and $\eta \to 0$, the ENBS $|k, 0, \mathrm{M}\rangle^-$ become Fock states $|k\rangle$ and $Q = -1$. Like the EBS, the ENBS also shows a significant amount of sub-Possonian statistics. From Figs. 1 and 2, we find that the sub-Poissonian statistics of both the EBS and ENBS are enhanced as k increases.

3.2. Squeezing Effects

From Eqs. (12) and (13), the expectation values of the operators a and a^2 on the EBS and ENBS are obtained as

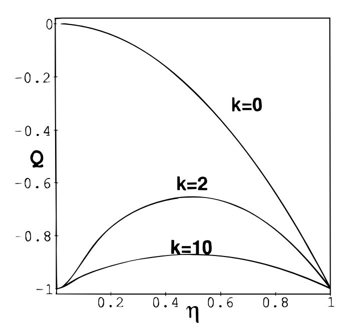

Fig. 1. Q parameter of the EBS as a function of η for different values of k ($M = 10$).

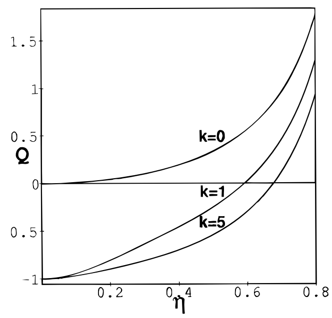

Fig. 2. Q parameter of the ENBS as a function of η for different values of k ($M = 3$).

$$\langle k, \eta, M|a|k, \eta, M\rangle = \sum_{n=k}^{M+k-1} \sqrt{n+1}\, D_n(k, \eta, M) D_{n+1}(k, \eta, M)$$

$$\langle k, \eta, M|a^2|k, \eta, M\rangle = \sum_{n=k}^{M+k-2} \sqrt{(n+2)(n+1)}\, D_n(k, \eta, M) \times D_{n+2}(k, \eta, M)$$

$${}^{-}\langle k, \eta, M|a|k, \eta, M\rangle^{-} = \sum_{n=k}^{\infty} \sqrt{n+1}\, D_n^{-}(k, \eta, M) D_{n+1}^{-}(k, \eta, M)$$

$${}^{-}\langle k, \eta, M|a^2|k, \eta, M\rangle^{-} = \sum_{n=k}^{\infty} \sqrt{(n+2)(n+1)}\, D_n^{-}(k, \eta, M) \times D_{n+2}^{-}(k, \eta, M) \tag{18}$$

Define the quadrature operators x (coordinate) and p (momentum) by

$$x = \tfrac{1}{2}(a^\dagger + a), \qquad p = \tfrac{i}{2}(a^\dagger - a) \tag{19}$$

In the present case $\langle a\rangle$ and $\langle a^2\rangle$ are real. Thus, the variances of x and p are

$$Var(x) = \tfrac{1}{4} + \tfrac{1}{2}(\langle a^\dagger a\rangle + \langle a^2\rangle - 2\langle a\rangle^2) \tag{20}$$

$$Var(p) = \tfrac{1}{4} + \tfrac{1}{2}(\langle a^\dagger a\rangle - \langle a^2\rangle)$$

Combining Eqs. (14), (18), and (20), we can study the squeezing effects of the EBS and ENBS.

In Fig. 3 we show the quantity $Var(x)$ of the EBS as a function of η for different values of k. The squeezing occurs in the quadrature component x over a wide range of parameters. As k increases, the range of squeezing becomes narrow and the degree of squeezing is reduced. Figure 4 gives the quantity $Var(p)$ of the ENBS as a function of η for different values of k. The ENBS exhibit squeezing in the quadrature component p. As for the EBS,

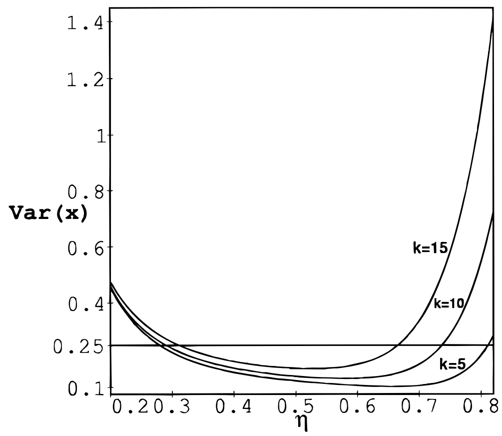

Fig. 3. The variance the quatrature component x of the EBS as a function of η for different values of k ($M = 10$).

the range of squeezing becomes narrow and the degree of squeezing is reduced as k increases.

4. CONCLUSIONS

We have introduced and investigated the EBS and ENBS and found the following:

1. The EBS and ENBS can be reduced to the Fock states and ECS. As intermediate states, both the EBS and ENBS interpolate between the Fock states and coherent states.

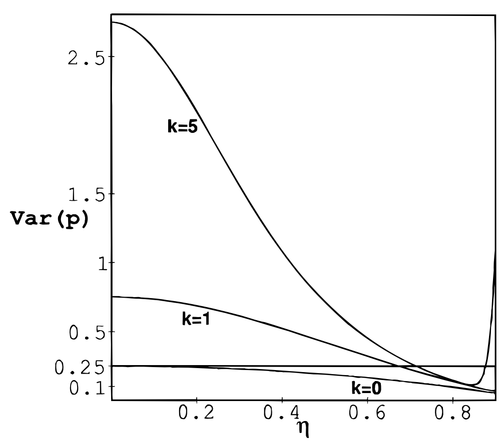

Fig. 4. The variance the quatrature component x of the ENBS as a function of η for different values of k ($M = 2$).

2. The EBS and ENBS are exactly normalized in terms of hypergeometric functions.

3. The EBS always shows sub-Poissonian statistics. The NBS always shows super-Poissonian statistics, but the ENBS can be sub-Poissonian. The sub-Poissonian statistics are enhanced as k increases for both the EBS and ENBS. In contrast to this, the squeezing properties of the two excited states are reduced and the range of squeezing becomes narrower as k increases. The squeezing occurs in the quadrature component x of the EBS and in the quadrature component p of the ENBS over a wide range of parameters.

The BS and NBS can be generated in some nonlinear processes [11, 12, 20]. Then, by a similar production mechanism of the ECS [1], if the excited atoms pass through a cavity, then, provided the field in the cavity is initially in a binomial state or a negative binomial state, one can produce the EBS and ENBS.

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REFERENCES

1. G. S. Agarwal and K. Tara (1991). *Phys. Rev. A* **43**, 492.
2. V. V. Dodonov, Ya. A. Korennoy, V. I. Man'ko, and Y. A. Moukhin (1996). *Quantum Semiclass. Opt.* **8**, 413.
3. Z. Zhang and H. Fan (1992). *Phys. Lett. A* **165**, 14.
4. D.-G. Welsch, M. Dakna, L. Knöll, and T. Opatrny, e-print quant-ph/9708018.
5. Zs. Kis, P. Adam, and J. Jansky (1994). *Phys. Lett. A* **188**, 16.
6. V. I. Man'ko and A. Wübsche (1997). *Quantum Semiclass. Opt.* **9**, 381.
7. G. S. Agarwal and K. Tara (1992). *Phys. Rev. A* **46**, 485.
8. G. N. Jones, J. Haight, and C. T. Lee (1997). *Quantum Semiclass. Opt.* **9**, 411.
9. M. Dakna, L. Knöll, and D.-G. Welsch (1998). *Opt. Commun.* **145**, 309.
10. D. Stoler, B. E. A. Saleh, and M. C. Teich (1985). *Opt. Acta* **32**, 345.
11. C. T. Lee (1985). *Phys. Rev. A* **31**, 1213.
12. G. Dattoli, J. Gallardo, and A. Torre (1987). *J. Opt. Soc. Am. B* **2**, 185.
13. A. V. Barranco and J. Roversi (1994). *Phys. Rev. A* **50**, 5233.
14. H. Y. Fan (1995). *Commun. Theor. Phys.* **24**, 483.
15. H. C. Fu and R. Sasaki (1996). *J. Phys. A: Math. Gen.* **29**, 5637.
16. K. Matsuo (1990). *Phys. Rev. A* **41**, 519.
17. V. Vourdas and R. F. Bishop (1995). *Phys. Rev. A* **51**, 2353.
18. R. Srinivasan and C. T. Lee (1996). *Phys. Lett. A* **218**, 151.
19. A. Joshi and A. S. F. Obada (1997). *J. Phys. A: Math. Gen.* **30**, 81.
20. H. C. Fu and R. Sasaki (1997). *J. Phys. Soc. Japan* **66**, 1989.
21. H. C. Fu and R. Sasaki (1997). *J. Math. Phys.* **38**, 3968.
22. S. M. Barnett (1998). *J. Mod. Opt.* **45**, 2201.
23. I. S. Gradshteyn and I. M. Ryzhik, *Tables of Integrals, Series and Products* (Academic, New York, 1965).